Mathematical and numerical modelling of the Wetropolis flood and rainfall demonstrator

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Talk outline

- Motivation: extreme rainfall and flooding
- Wetropolis flood demonstrator: background and description
- ► Mathematical and numerical modelling of rivers: open-channel flow
- ▶ Simulations: the Wetropolis 'live' dashboard
- ► Current/next steps: ...



Inspiration for Wetropolis: the Boxing Day 2015 floods of the River Aire in Leeds

Motivation: high-impact weather

E.g., Storm Ciara (Feb. 2020): from $\frac{1}{1}$ from





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E.g., Storm Ciara (Feb. 2020): from numerical weather prediction to flooding (in Leeds)



Source: Met Office and LeedsLive.

Urban flooding is a major hazard worldwide, brought about by intense rainfall and often exacerbated by the built environment.

Flood mitigation requires accurate predictions (good models + data) as well as effective communication and engagement of stakeholders and the public.

Introducing Wetropolis (Est. 2016)

- ▶ interactive table-top model of extreme rainfall and flooding (outreach project)
- conceptualises many important aspects of the science of flooding and extreme events in a way that is accessible to and directly engages the public...





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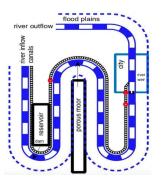




Also of interest from a (fluid dynamical) modelling perspective: (i) design model and (ii) testbed for predictive modelling and data assimilation.

Introducing Wetropolis: design and set-up

GOAL (2016): to demonstrate random extreme rainfall and flood events in a physical model on reduced spatial and temporal scales.



Components:

- a winding river channel with parallel canal,
- a reservoir for water storage,
- a porous groundwater cell (analogous to a moor)
- and random daily rainfall.

Water enters the river channel in four places:

- (i) the upstream inflow;
- (ii) overflow from the reservoir;
- (iii) overflow from the groundwater cell; and
- (iv) via the canal in the city.

The river bed is sloping down (uniformly with gradient 1 in 100); different channel cross-sectional areas in floodplain regions and urban/city region (more later on).

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Rainfall is supplied randomly:

- in space at four locations (reservoir, moor, reservoir and moor, or nowhere)
- in time at four rainfall rates corresponding to 10%, 20%, 40%, or 90% of a Wetropolis day (wd)
- joint probabilities (rain amount times rain location) determined daily via two asymmetric Galton boards (16 possible outcomes)

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The most extreme daily rainfall event thus involves rainfall on both moor and reservoir for 90% of a Wetropolis day with probability 7/256=0.027...; i.e., we must wait on average $256/7\approx36$ wds for an 'extreme' rainfall event.

Wetropolis: modelling

GOAL (2016): to demonstrate random extreme rainfall and flood events in a physical model on reduced spatial and temporal scales.

Achieved! How? Numerical design model to determine the relevant time and length scales prior to construction: crude, inexpensive but sufficient for rapid exploration of design choices.

Outcome: pump flow-rates determined, length of channel, Wetropolis day = 10 seconds, etc... lots of fun!

Recall: we must wait on average $256/7 \approx 36$ wds for an 'extreme' rainfall event, so just over 6 minutes in real-time... important concept of return period.

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GOAL (2020): to develop the hydrodynamic modelling further and improve the visualisation of its output, with a view to conducting real-time simulations with data assimilation.

Why?

- to enhance the outreach experience with a live display of the (real-time) numerical simulations in tandem with the physical set-up
- to investigate potential issues that arise when combining (imperfect) models and data in an idealised environment

Open-channel flow

- Examples in our environment: natural (e.g., rivers and streams) and man-made waterways (e.g., conduits, canals, drainage and sewer systems).
- ► Fluid flows through channels of varying geometries with a free surface.
- Fluid mechanical modelling of such flows is termed hydraulic modelling
- Unsteady open-channel flow is typically modelled using the St. Venant equations in one (along-channel) spatial dimension with along-channel coordinate.

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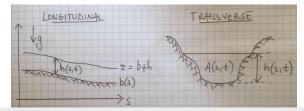
St. Venant equations

Equations of motion: hierarchy

Navier-Stokes $\xrightarrow{\text{depth-}}_{\text{average}}$ 2D SWEs $\xrightarrow{\text{area-}}_{\text{average}}$ 1D St. Venant

Assumptions:

- flow is hydrostatic, i.e., horizontal length and velocity scales well exceed their vertical counterparts such that vertical fluid accelerations are negligible;
- flow is one-dimensional, i.e., the transverse free surface is horizontal and the velocity is approximately uniform in a cross-section;
- channel curvature is small and the bed slope is small,
- sediment and bed motion are neglected on the timescales considered.



St. Venant equations

Model variables: cross-sectional area A=A(s,t) and velocity u=u(s,t), both functions of the along-channel spatial coordinate s and time t.

Continuity:
$$\partial_t A + \partial_s (Au) = S_A(s,t),$$
 (1a)

Velocity:
$$\partial_t u + u \partial_s u + g \partial_s h - g \left(S_o - S_f \right) = 0,$$
 (1b)

where:

- ► $S_A(s,t)$ is the mass source term [units m²s⁻¹],
- $ightharpoonup S_o = -\partial_s b$ is the bed slope [dimensionless], where b = b(s) is the bathymetry,
- $ightharpoonup S_f$ is the friction term [dimensionless],
- ▶ and g is the gravitational acceleration [units ms^{-2}].

Note that water depth h=h(s,t) appears in (1b) and is known via the (invertible) function h=h(A(s,t),s); it depends explicitly on both A=A(s,t) and s.

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The Manning relation for friction:

$$S_f = \frac{C_m^2}{R^{4/3}} u|u|, (2)$$

where $R=R(h)=\frac{\text{wet area}}{\text{wetted perimeter}}$ [units m] is the hydraulic radius and C_m is the Manning friction coefficient.

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$$\frac{\partial_t u + u \partial_s u + g \partial_s h - g \left(S_o - S_f \right) = 0}{\partial_t u + u \partial_s u + g \partial_s h - g \left(S_o - S_f \right)} = 0 \quad \Longrightarrow \quad u = \frac{R^{2/3}}{C_m} \sqrt{-\partial_s b}, \tag{3}$$

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Substituting the kinematic velocity into the continuity equation, we obtain the kinematic model:

$$\partial_t A + \partial_s \left(\frac{AR^{2/3}}{C_m} \sqrt{-\partial_s b} \right) = S_A(s, t).$$
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To perform accurate predictions of the hydrodynamics in Wetropolis, more advanced models, e.g., the St. Venant equations, are likely required, in combination with an efficient data assimilation algorithm to constrain the model.

St. Venant equations: conservative form

Combining the continuity and velocity equations, the evolution equation for the momentum, or discharge, Q=Au [units ${\rm m}^3{\rm s}^{-1}$] can be derived:

$$\partial_t(Au) + \partial_s\left(Au^2\right) + \underline{gA\partial_s h} = -g\left(A\partial_s b + \frac{C_m^2 Au|u|}{R(A)^{4/3}}\right) + uS_A(s,t). \tag{5}$$

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Noting that the pressure term (underlined above) is in nonconservative form, it can be expressed as follows:

$$A\partial_s h = \partial_s (A h) - h\partial_s A. \tag{6}$$

The Ah term is now in conservative form and the derivative no longer acts on h directly but on the dependent variable A=A(s,t).

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The conservative form of the St. Venant equations can be re-expressed as:

$$\partial_t A + \partial_s (Au) = S_A(s,t),$$
 (7a)

$$\partial_t(Au) + \partial_s \left(Au^2 + gAh\right) - \underline{gh\partial_s A} = -g\left(A\partial_s b + \frac{C_m^2 Au|u|}{R^{4/3}}\right) + uS_A(s,t), \quad (7b)$$

where the nonconservative product has been underlined.

St. Venant: hyperbolic?

St. Venant equations in vector form:

$$\partial_t U + \partial_s F + G \partial_s U = S, \tag{8}$$

with vector of unknowns $\pmb{U} = \pmb{U}(s,t)$, flux \pmb{F} , nonconservative product matrix \pmb{G} and source/sink vector \pmb{S} defined by

$$\mathbf{U} = (A, Au)^T, \quad \mathbf{F} = \mathbf{F}(\mathbf{U}) = (Au, Au^2 + g h A)^T,$$

$$\mathbf{G} = \mathbf{G}(\mathbf{U}) = \begin{pmatrix} 0 & 0 \\ -gh & 0 \end{pmatrix}, \quad \mathbf{S} = \mathbf{S}(\mathbf{U}) = \left(S_A, -gA\partial_s b - g\frac{C_m^2 Au|u|}{R^{4/3}} + uS_A\right)^T.$$
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The Jacobian matrix $m{J}=\partial m{F}/\partial m{U}+m{G}$ for system (8) is then given by

$$J = \begin{pmatrix} 0 & 1 \\ -u^2 + gA\partial h/\partial A & 2u \end{pmatrix}. \tag{10}$$

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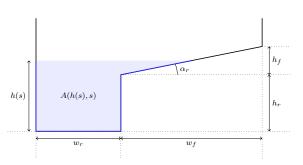
and its eigenvalues of J are

$$\lambda_{\pm} = u \pm \sqrt{gA \,\partial h/\partial A}.\tag{11}$$

If $\partial h/\partial A$ is positive, the eigenvalues are real and therefore the system is hyperbolic. There exist novel numerical methods for integrating (nonconservative) hyperbolic systems of PDEs ... e.g., FV or discontinuous Galerkin FEM.

Wetropolis: channel geometry I

Schematic of the channel geometry with a one-sided sloping floodplain: the cross-sectional area A=A(h,s) (blue shaded area) with associated wetted perimeter W=W(h) (thick blue line), in this instance for $h_r < h < h_r + h_f$. When $h < h_r$, the water flows in the rectangular channel of width w_r .



Water depth h as a function of cross-sectional area A and along-channel coordinate s:

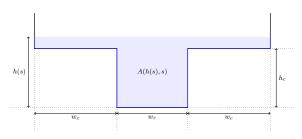
$$h(A,s) = \left\{ \begin{array}{ll} A/w_r & \text{when } A < A_1; \\ h_r - w_r \tan \alpha_r + \sqrt{w_r^2 \tan \alpha_r^2 + 2(A - w_r h_r) \tan \alpha_r} & \text{when } A < A_2; \\ \frac{\left(A + w_f \left(h_r + \frac{1}{2}h_f\right)\right)}{w_r + w_f} & \text{when } A > A_2; \end{array} \right.$$

with $A_1=w_rh_r$ and $A_2=(h_r+h_f)(w_r+w_f)-w_f(h_r+\frac{1}{2}h_f)$. Note that $\partial h/\partial A>0$.

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Wetropolis: channel geometry II

Schematic of the channel geometry in the city: the cross-sectional area A=A(h,s) (blue shaded area) with associated wetted perimeter W=W(h) (thick blue line), in this instance for $h>h_c$ (in flood). When $h< h_c$, the water flows in the rectangular channel of width w_c .



Water depth h as a function of cross-sectional area A and along-channel coordinate s:

$$h(A,s) = \begin{cases} A/w_r & \text{when } A < A_c; \\ \frac{A+2w_ch_r}{w_r + 2w_c} & \text{when } A > A_c; \end{cases}$$

with $A_c = w_r h_c$. Note that $\partial h/\partial A > 0$.

The full Wetropolis system of river, moor, canals, and reservoir, all coupled together with weir relations, for the unknowns A(s,t), u(s,t), $h_m(y,t)$, $h_{res}(t)$, $h_{1c}(t)$, $h_{2c}(t)$ and $h_{3c}(t)$ is as follows:

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River:
$$\begin{cases} \partial_t A + \partial_s (Au) = S_A \\ \partial_t (Au) + \partial_s \left(Au^2\right) + gA\partial_s h = -g\left(A\partial_s b + \frac{C_m^2 Au|u|}{R^{4/3}}\right) + uS_A & \text{on } s \in [0, L] \\ \text{with } h = h(A(s,t)), \quad h(s,0) = h_0(s), \quad u(s,0) = u_0(s), \\ \text{and } S_A(t) = (1 - \gamma)Q_{res}(t)\delta(s - s_{res}) + Q_{moor}(t)\delta(s - s_{moor}) + Q_{1c}(t)\delta(s - s_{1c}) \end{cases}$$

$$(37a)$$

$$\text{Moor:} \quad \partial_t(w_v h_m) - \alpha g \partial_y \left(w_v h_m \partial_y h_m\right) = \frac{w_v R_{moor}(t)}{m_{por} \sigma_e} \quad \text{on} \quad y \in [0, L_y]$$

with
$$\partial_t h_m|_{y=L_y} = 0$$
, $h_m(0, t) = h_{3c}(t)$, $h_m(y, 0) = h_{m0}(y)$ (37b)

Reservoir:
$$w_{res}L_{res}\frac{dh_{res}}{dt} = w_{res}L_{res}R_{res}(t) - Q_{res}$$
, with $h_{res}(0) = h_{r0}$ (37c)

Canal-1:
$$w_c(L_{1c} - L_{2c}) \frac{dh_{1c}}{dt} = Q_{2c} - Q_{1c}$$
, with $h_{1c}(0) = h_{10}$ (37d)

Canal-2:
$$w_c(L_{2c} - L_{3c}) \frac{dh_{2c}}{dt} = Q_{3c} - Q_{2c}$$
, with $h_{2c}(0) = h_{20}$ (37e)

Canal-3:
$$w_c L_{3c} \frac{dh_{3c}}{dt} = \gamma Q_{res} - Q_{3c}$$
, with $h_{3c}(0) = h_{30}$, (37f)

Influxes:
$$Q_{1c} = C_f \sqrt{g} w_c \max(h_{1c} - P_{1w}, 0)^{3/2}$$
 (37g)

$$Q_{2c} = C_f \sqrt{g} w_c \max(h_{2c} - P_{2w}, 0)^{3/2}$$
(37h)

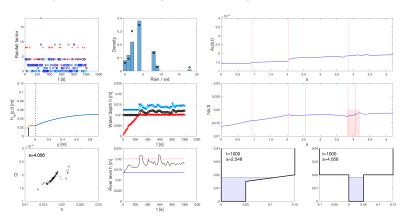
$$Q_{3c} = C_f \sqrt{g} w_c \max(h_{3c} - P_{3w}, 0)^{3/2}$$
(37i)

$$Q_{moor} = \frac{1}{2} m_{por} \sigma_e w_v \alpha g(\partial_y h_m)^2 |_{y=0}$$
(37j)

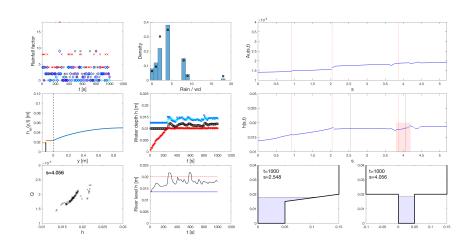
$$Q_{res} = C_f \sqrt{g} w_{res} \max(h_{res} - P_{wr}, 0)^{3/2}$$
(37k)

Simulations

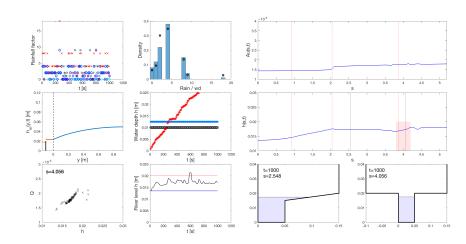
- ► Taster: https://github.com/tkent198/hydraulic_wetro#taster
- ► Full system: https://github.com/tkent198/hydraulic_wetro#preliminary-simulations
- Many more online, including code and running instructions...



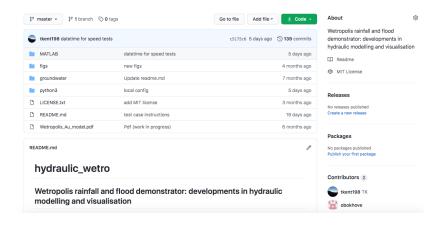
Wetropolis dashboard: control via reservoir storage?



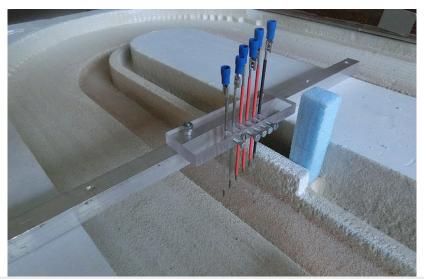
Wetropolis dashboard: control via reservoir storage?



GitHub repository: hydraulic_wetro



Future: WetroDA



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Thanks very much for your attention ... any questions?

References:

- Bokhove, O., Hicks, T., Zweers, W. and Kent, T. (2020): Wetropolis extreme rainfall and flood demonstrator: from mathematical design to outreach. *Hydrol. Earth Syst. Sci.*, 24, 2483–2503. [Selected as a journal highlight May 2020.]
- Kent, T. (2020): Wetropolis rainfall and flood demonstrator: developments in hydraulic modelling and visualisation [hydraulic_wetro], Open-source code and manual: https://github.com/tkent198/hydraulic_wetro.
- Kent, T., Bokhove, O., Zweers, W. (2017): 'Wetropolis' flood demonstrator. Outreach project report, Maths Foresees.

Research: interests

Mathematical and statistical modelling of atmospheric and environmental phenomena, including:

- geophysical fluid dynamics;
- numerical methods (in particular for hyperbolic problems);
- hydraulic and shallow water-type modelling;
- numerical weather prediction and (ensemble-based) data assimilation;
- flood modelling and mitigation;
- statistical downscaling and bias correction.