An idealised fluid model for convective-scale NWP: dynamics and data assimilation

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NWP and DA: from large- to convective-scale

Data Assimilation (DA) attempts to provide the optimal initial conditions for the forecast model

- yields best estimate of the state of the atmosphere and its uncertainty using a combination of forecast and observational information (and taking into account their respective uncertainties)
- optimality of the initial state is crucial in such a highly nonlinear system with limited predictability
- a great deal of attention is focussed on improving observing systems and assimilation algorithms

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High-resolution (convective-scale) NWP models are becoming the norm

- more dynamical processes such as convection, cloud formation, and small-scale gravity waves, are resolved explicitly/partially
- increasing resolution is not a panacea: 'grey zone' presents many problems

Using idealised models

DA techniques need to evolve in order to keep up with the developments in high-resolution NWP

- breakdown of dynamical balances (e.g., hydrostatic and semi/quasi-gestrophic) at smaller scales
- strongly nonlinear processes associated with convection and moisture/precipitation
- move towards ensemble-based methods: flow-dependent errors, development of ensemble prediction systems...

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It may be unfeasible, and indeed undesirable, to investigate the potential of DA schemes on state-of-the-art NWP models. Instead idealised models can be employed that:

- capture some fundamental processes
- are computationally inexpensive to implement

'Toy' models:

- Lorenz (L63, L95, L2005, ...)
- BV/QG models
- simplified NWP models

Using idealised models: approach

1. Introduce a physically plausible idealised model and investigate numerically.
   - based on the rotating shallow water equations (SWEs) and the model of Würsch and Craig (2014) for simplified cumulus cloud dynamics
   - compare dynamics of the modified model to those of the classical shallow water theory

2. Ensemble-based DA - relevant for convective-scale NWP?
   - algorithm: perturbed obs. EnKF
   - for meaningful experiments:
     - dynamics: relevant time- and length-scales
     - DA: "tuning" the observing system and ensemble configuration
     - exploiting the model's strong non-linearity

An idealised model of convective-scale NWP

Moist convection... *is many things* (Stevens 2005)

- Manifest as clouds, it comprises a variety of regimes spanning a vast range of spatial and temporal scales, with diverse and nonlinear physical processes in each regime...
- state-of-the-art numerical models of the atmosphere struggle with their treatment of moist convection
- one can seek to represent some of the fundamental processes and aspects of moist convection in a relatively simple modelling environment.

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Idealised model: concept

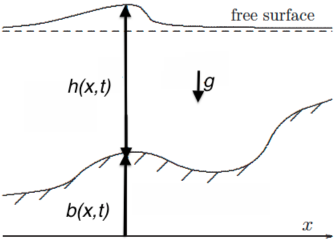
- *"It is almost as if the fluid is magically transformed into another form once it crosses a certain threshold..."* (Stevens again)
- "*moist convection can in many instances be thought of as a two-fluid problem, where one fluid (unsaturated air) can transform itself into another (saturated air) simply through vertical displacement.*"
- Würsch and Craig (2014) model: the single–layer shallow water equations are modified when the height of the fluid crosses certain thresholds
- the behaviour of the flow is transformed from the standard shallow water dynamics to a simplified representation of convection, with associated precipitation effects

1. SWEs

$1.5D$ rotating SWEs on an f-plane with no variation in the y-direction ($\partial_y = 0$):

$$\partial_t h + \partial_x (hu) = 0,$$
$$\partial_t (hu) + \partial_x (hu^2 + p(h)) - fhv = -gh\partial_x b,$$
$$\partial_t (hv) + \partial_x (huv) + fhu = 0,$$

where $p(h)$ is an effective pressure: $p(h) = \frac{1}{2} g h^2$.



- derived from the inviscid, incompressible, hydrostatic equations of motion
- often used in meteorology for idealised models of atmospheric circulation
- admit gravity waves (important for convection) and, in the presence of rotation, have scale interactions (fast gravity wave modes and slow geostrophic modes)
- admit discontinuous solutions, akin to the propagation of atmospheric fronts

1. SWEs: an extension

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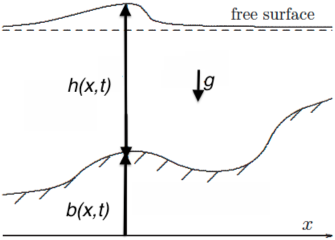

Aim: modify the SWEs to include more complex dynamics relevant for the 'convective-scale', extending the model employed by Würsch and Craig (2014)

- convective updrafts - artificially mimic conditional instability (positive buoyancy)
- idealised representation of precipitation ('rain' mass fraction), including source and sink
- switches for the onset of convection and precipitation - realistic (and highly nonlinear) features of operational NWP models

Modified SWEs

Ingredients:

- two threshold heights $H_c < H_r$: when fluid exceeds these heights, different mechanisms kick in and alter the classical SW dynamics
- modifications to the effective pressure gradient (equivalently, geopotential gradient) in the momentum equation
- extra equation for the conservation of model 'rain' to close the system

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- extra equation for the conservation of model 'rain' to close the system

$$
\begin{aligned}
&\partial_t h + \partial_x (hu) = 0, \\
&\partial_t (hu) + \partial_x (hu^2 + P) + hc_0^2 \partial_x r - fhv = -Q \partial_x b, \\
&\partial_t (hv) + \partial_x (huv) + fhu = 0, \\
&\partial_t (hr) + \partial_x (hur) + h\widetilde{\beta} \partial_x u + \alpha hr = 0,
\end{aligned}
$$

$$
\text{where } P = P(h;b) = \begin{cases} p(H_c - b), & \text{for } h + b > H_c, \\ p(h), & \text{otherwise,} \end{cases}
$$

$$
Q = Q(h;b) = \begin{cases} p'(H_c - b), & \text{for } h + b > H_c, \\ p'(h), & \text{otherwise,} \end{cases}
$$

$$
\text{and } \widetilde{\beta} = \begin{cases} \beta, & \text{for } h + b > H_r,\ \partial_x u < 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Some theoretical aspects

Eigenanalysis: shallow water systems are hyperbolic, and can thus be solved via a range of numerical recipes for hyperbolic systems. What about the modified system?

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) + \boldsymbol{G}(\boldsymbol{U})\partial_x \boldsymbol{U} + \boldsymbol{S}(\boldsymbol{U}) = 0,$$

Hyperbolicity determined by eigenstructure (all eigenvalues must be real). Eigenvalues of the system are determined by the Jacobian matrix $\partial \boldsymbol{F}/\partial \boldsymbol{U} + \boldsymbol{G}(\boldsymbol{U})$:

$$\lambda_{1,2} = u \pm \sqrt{\partial_h P + c_0^2 \widetilde{\beta}}, \quad \lambda_{3,4} = u.$$

Since $P(h; b)$ is non-decreasing and $\widetilde{\beta}$ non-negative, the eigenvalues are real. Hence, the modified SW model is (weakly) hyperbolic.

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Wave speeds: waves travelling through (saturated) regions of convection slow down

- $h + b < H_c$: $\partial_h P = gh$, $\widetilde{\beta} = 0$ implies standard eigenvalues $\lambda_{1,2} = u \pm \sqrt{gh}$
- $H_c < h + b < H_r$: $\partial_h P = 0$, $\widetilde{\beta} = 0$ implies modified eigenvalues $\lambda_{1,2} = u$
- $H_r < h + b$: $\partial_h P = 0$ and $\widetilde{\beta} = \beta$ implies modified eigenvalues $\lambda_{1,2} = u \pm \sqrt{c_0^2 \beta}$

Numerics

Methodology:

- Rhebergen et al. (2008): a novel discontinuous Galerkin (DG) finite element framework for non-conservative hyperbolic system of PDEs, deals robustly with high nonlinearity and non-conservative products, $\boldsymbol{G}(\boldsymbol{U})\partial_x \boldsymbol{U}$
- combine with the scheme of Audusse et al. (2004) to discretise topography: maintains well-balancedness and preserves non-negativity of fluid depth and rain variable

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Experiments: based on (i) a Rossby adjustment scenario, and (ii) non-rotating flow over topography. To illustrate the effect that exceeding the threshold heights $H_c < H_r$ has on the dynamics, a hierarchy of model 'cases' is employed:

- Case I: $h + b < H_c$ always (effectively setting $H_c, H_r \to \infty$). The model reduces to standard RSWEs if $hr = 0$ initially.
- Case II: $h + b < H_r$ always, but may exceed H_c. This is considered a 'stepping stone' to the full model to isolate the effect of the first threshold exceedance.
- Case III: $h + b$ may exceed both H_c, H_r. This is the idealised fluid model with convection and rain processes to be used for convective-scale DA experiments.

Solve non–dimensionalised equations, with 2 prognostic parameters: Ro and Fr.

(i) Rossby adjustment scenario

- rotating flow $(\mathrm{Ro} = 0.1, \mathrm{Fr} = 1)$ with flat bottom topography $(b \equiv 0)$
- the free surface height h is disturbed from its rest state by a transverse jet, i.e., fluid with an initial constant height profile is subject to a localised v-velocity distribution
- to adjust to this initial momentum imbalance, the height field evolves rapidly, emitting inertia gravity waves and shocks that propagate out from the jet and eventually reach a state of geostrophic balance.

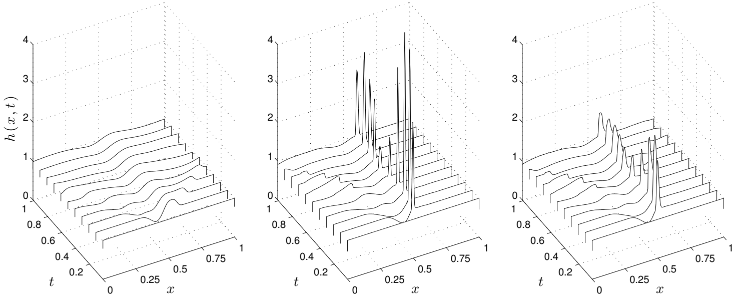
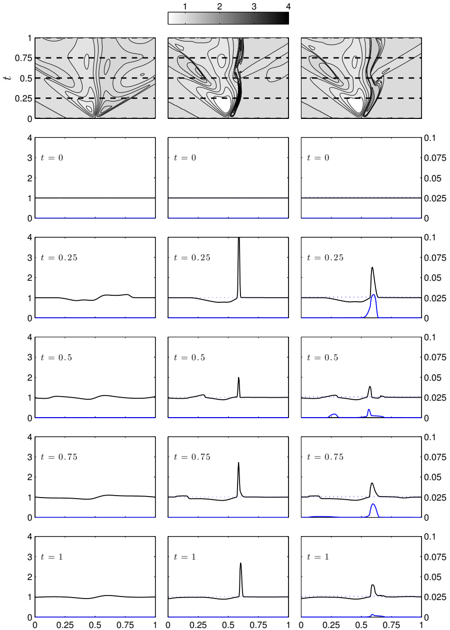


Figure : Time evolution of the height profile: case I (left), II (middle), III (right).

Snapshots of $h(x,t)$ and $r(x,t)$ for the Rossby adjustment process with initial transverse jet $(\mathrm{Ro} = 0.1, \mathrm{Fr} = 1)$:

- case I (left), II (middle), and III (right)
- Top row: Hovmöller plots for h
- Subsequent rows: profiles of h (black line; left axis) and r (blue line; right axis) at different times denoted by the dashed lines in the top row



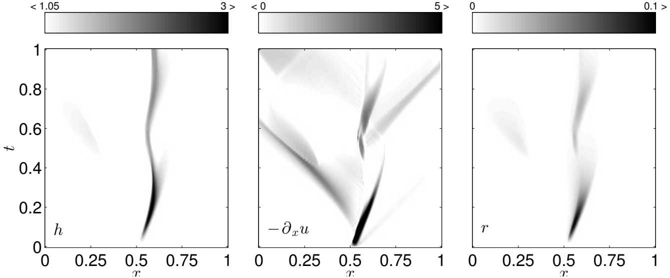
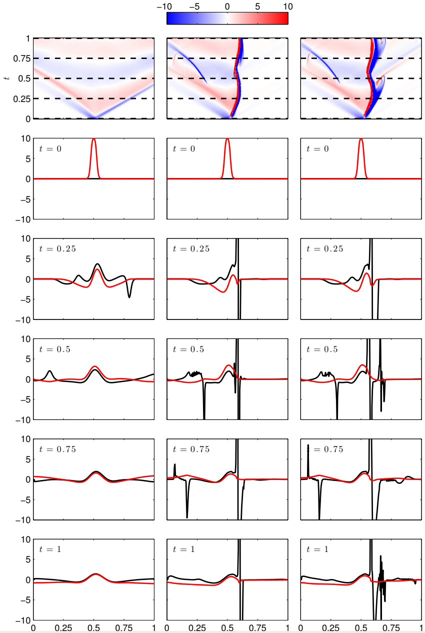


Figure : Hovmöller plots for the Rossby adjustment process with initial transverse jet $(\mathrm{Ro} = 0.1, \mathrm{Fr} = 1)$, highlighting the conditions for the production of rain: case III. From left to right: $h > H_r$, $-\partial_x u > 0$, and $r(x, t)$.

- Top row: Hovmöller diagram plotting the evolution of the measure of departure from geostrophic balance $g\partial_x h - fv$: light shading denotes regions close to geostrophic balance
- Subsequent rows: profiles of fv (red) and $g\partial_x h$ (black) at different times denoted by the dashed lines in the top row
- case I (left), II (middle), and III (right)



(ii) Flow over topography

Consider non-rotating flow (no transverse velocity) over a parabolic ridge:

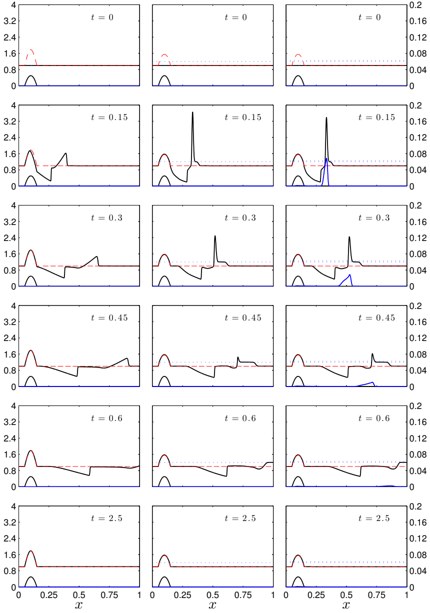
$$b(x) = \begin{cases} b_c \left(1 - \left(\frac{x - x_p}{a}\right)^2\right), & \text{for } |x - x_p| \leq a; \\ 0, & \text{otherwise;} \end{cases}$$

where b_c is the height of the hill crest, a is the hill width parameter, and x_p its location in the domain.

- shallow water flow over topography has been extensively researched (see, e.g., Baines 1998)
- often used as a test case in numerical studies owing to the range of dynamics (dependent on Froude number Fr), including shocks, and the existence of analytical non-trivial steady state solutions
- supercritical flow $\mathrm{Fr} > 1$: the fluid depth increases over the ridge (as opposed to subcritical flow ($\mathrm{Fr} < 1$) in which the depth decreases over the ridge) and a shock wave propagates at a height above the rest depth to the right of the ridge

Flow over topography ($\mathrm{Fr} = 2$, $b_c = 0.5$, $a = 0.05$, and $x_p = 0.1$):

- profiles of $h + b$, b (black; left y-axis), rain r (blue; right y-axis), and exact steady-state solution for $h + b$ (red dashed) at different times
- case I (left), II (middle), and III (right)
- dotted lines denote the threshold heights for convection and rain $H_c < H_r$



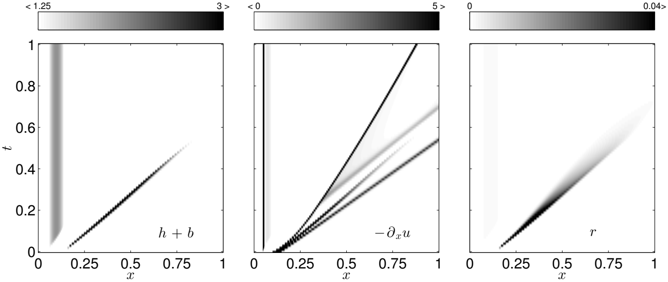
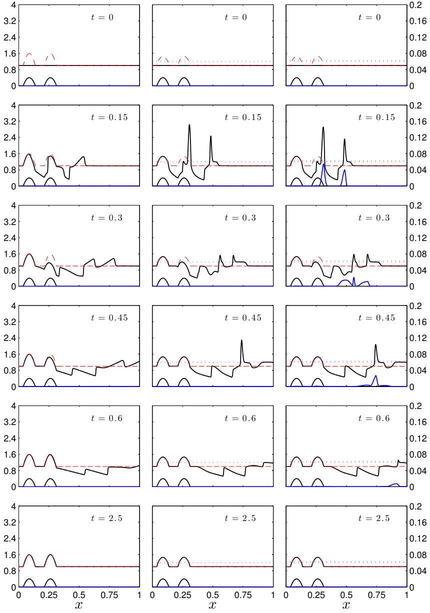


Figure : Hovmöller plots for flow over topography $(\mathrm{Fr} = 2)$, highlighting the conditions for the production and subsequent evolution of rain: case III. From left to right: $h + b > H_r$, $-\partial_x u > 0$, and $r(x, t)$.

Flow over topography ($\mathrm{Fr} = 2$, $b_c = 0.4$, $a = 0.05$, $(x_{p_1}, x_{p_2}) = (0.0875, 0.2625)$:

- profiles of $h + b$, b (black; left y-axis), rain r (blue; right y-axis), and exact steady-state solution for $h + b$ (red dashed) at different times
- case I (left), II (middle), and III (right)
- dotted lines denote the threshold heights for convection and rain $H_c < H_r$



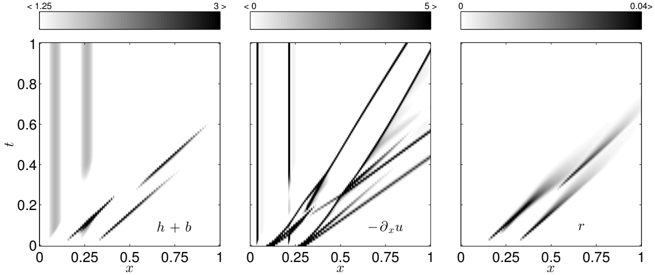


Figure : Hovmöller plots for flow over two ridges ($\mathrm{Fr} = 2$), highlighting the conditions for the production and subsequent evolution of rain: case III. From left to right: $h + b > H_r$, $-\partial_x u > 0$, and $r(x, t)$.

Dynamics: summary

- changes to the dynamics are brought about by the exceedance of two threshold heights H_c and H_r, akin to (i) the level of free convection, and (ii) the onset of precipitation
- when the fluid exceeds these heights, the classical shallow water dynamics are altered to include a representation of conditional instability (leading to a convective updraft) and idealised moisture transport with associated downdraft and precipitation effects
- the model reduces exactly to the standard SWEs in non-convecting, non-precipitating regions

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- the model reduces exactly to the standard SWEs in non-convecting, non-precipitating regions
- the model also exhibits important aspects of convective-scale dynamics relating to the disruption of large-scale balance principles
  - Rossby adjustment scenario illustrates the breakdown of geostrophic balance in the presence of convection and precipitation
  - breakdown of hydrostatic balance is implicity enforced by the modified pressure when the level of free convection is exceeded
- able to simulate other features related to convecting and precipitating weather systems, such as the initiation of daughter cells away from the parent cell by gravity wave propagation, and convection downstream from an orographic ridge.

Using idealised models: approach

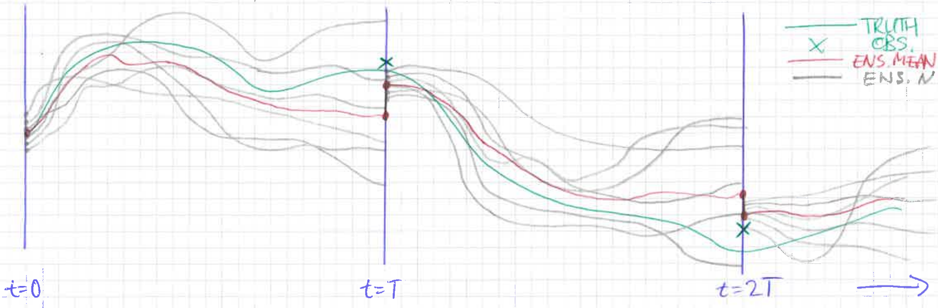
1. Describe a physically plausible idealised model and implement numerically ✓
   - based on the rotating shallow water equations (SWEs) and the model of Würsch and Craig (2014) for simplified cumulus cloud dynamics ✓
   - compare dynamics of the modified model to those of the classical shallow water theory ✓

2. Ensemble-based DA - relevant for convective-scale NWP?
   - algorithm: perturbed obs. EnKF
   - for meaningful experiments:
     - dynamics: relevant time- and length-scales
     - DA: "tuning" the observing system and ensemble configuration
     - exploiting the model's strong non-linearity

Houtekamer and Zhang (2016): "*The frontier of data assimilation is at the high spatial and temporal resolution, where we have rapidly developing precipitating systems with complex dynamics*".

ENSEMBLE KALMAN FILTER

- Model state $\underline{x} \in \mathbb{R}^n$; error covariance $P^f \in \mathbb{R}^{n \times n}$
- Observations $\underline{y} \in \mathbb{R}^p$, $(n >> p)$; error cov. $R \in \mathbb{R}^{p \times p}$
- Obs. operator $H : \mathbb{R}^n \longrightarrow \mathbb{R}^p$; $\underline{y} = H(\underline{x}^t) + \underline{\varepsilon}^o$



- Analysis: $\underline{x}^a = \underline{x}^f + \underbrace{P^f H^T (H P^f H^T + R)^{-1}}_{=:K} (\underline{y} - H \underline{x}^f)$

$$\Rightarrow \underline{x}^a = K \underline{y} + (I - KH) \underline{x}^f$$

2. Ensemble-based DA for idealised models

GOAL: show that the idealised fluid model provides an interesting testbed for investigating DA algorithms in the presence of convection and precipitation.

I.e., demonstrate a well-tuned idealised ensemble forecast-assimilation system using the model and elucidate its relevance for convective-scale NWP

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1. Forecast model: $\boldsymbol{x}(t) = \mathcal{M}(\boldsymbol{x}(t-1))$
   - non-linear discrete model $\mathcal{M}$ provided by the idealised fluid model described in part one
   - (imperfect) twin-model setting: forecasts and nature run obtained by integrating $\mathcal{M}$ at different resolutions
2. Assimilation algorithm: perturbed obs. EnKF
   - tune the observing system (e.g., observation error, density) and ensemble configuration (e.g., inflation (additive and multiplicative), localisation)
   - seek to mimic (where possible) characteristics relevant for NWP (e.g., observational influence, update frequency)

2. Ensemble-based DA for idealised models

- What constitutes a well-tuned experiment?
  - RMSE of the ensemble mean vs. ensemble spread: a well-configured ensemble (i.e., sufficiently spread) is crucial to providing an adequate estimation of forecast error
  - observational influence diagnostic (after Cardinali et al. (2004)): most weight comes from the background ($\sim 82\%$ in global NWP, 60-85% (?) in high-res. NWP)

$$OI = \frac{tr(\mathbf{HK})}{p}$$

  - others: CRPS, error-doubling time...

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- How is this achieved?
  - addressing the rank / sampling issues due to small ensemble size:
    - ensemble inflation techniques: multiplicative (for underestimation of variances) and/or additive (for model error)
    - localisation ($\boldsymbol{P}^f_{loc} \leftarrow \rho_{loc} \circ \boldsymbol{P}^f$) applied to damp spurious long-range correlations
  - tuning the observing system: what to observe? how often? with how much noise?

Idealised EnKF experiments: flow over topography

Dynamics: time- and length-scales

- non-dimensional parameters: $\mathrm{Ro} = \infty$, $\mathrm{Fr} = 1.1$ (supercritical)
- length of domain ~ 500 km: 200 gridcells implies forecast resolution of ~ 2.5km and $\dim(x) = 600$
- topography: superposition of sinusoidal waves in part of the domain (á la the Pennines!)
- BCs: periodic

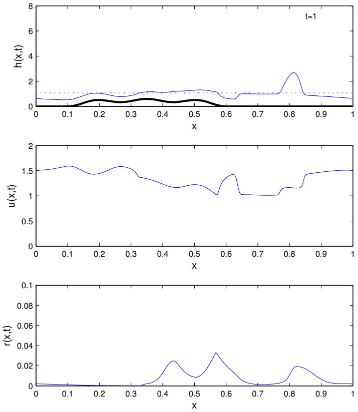
Assimilation: set-up

- hourly cycling for 48hrs (NOTE: 1 non-dim T $\sim$ 7hrs)
- fix the number of ensemble members $N = 40$ and permute through various tuning parameters relating to the filter configuration and observing system:
  - obs. noise $\boldsymbol{\sigma}_o = (\sigma_h, \sigma_u, \sigma_v, \sigma_r)$
  - obs density (e.g., observe every 20 gridcells ~ 50km)
  - localisation length-scale
  - inflation factors

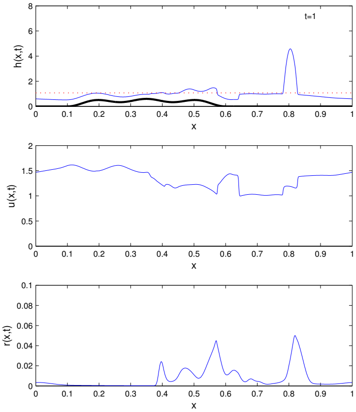
Imperfect model

Ensemble Kalman filter: imperfect model setting - nature run at higher resolution...

Forecast: 200 cells ("smoother features")



Nature: 800 cells ("sharper features")



Additive inflation

- additive inflation consists of adding random Gaussian perturbations $\boldsymbol{\eta}_j \sim \mathcal{N}(0, \gamma \boldsymbol{Q})$ during the forecast step:

$$\boldsymbol{x}_j(t) = \mathcal{M}(\boldsymbol{x}_j(t-1)) + \boldsymbol{\eta_j}, \quad j = 1, ..., N$$

- acts in some sense as a lower bound for the forecast error, thus preventing filter divergence
- addition of random Gaussian noise at each analysis time tends to erase the non-Gaussian higher moments non-linear error growth may have generated

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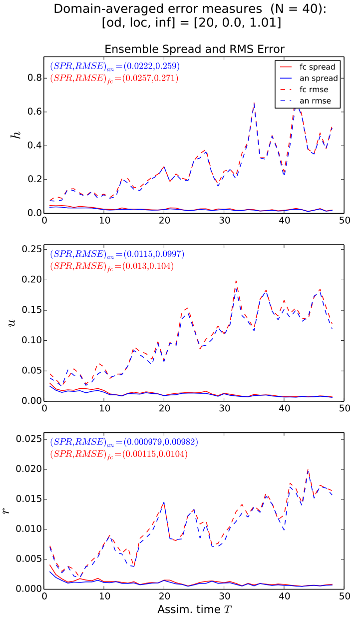
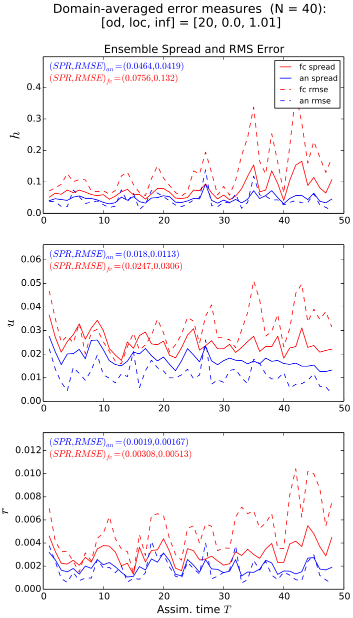
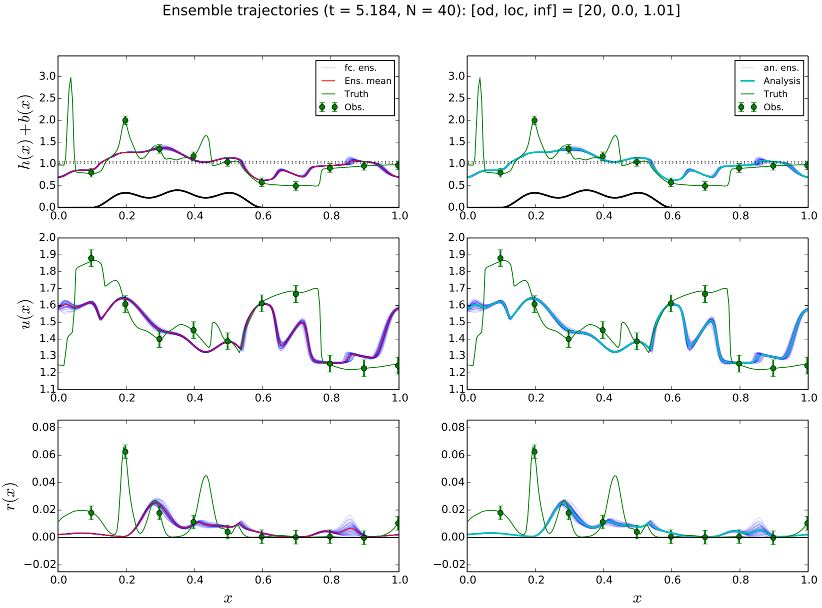
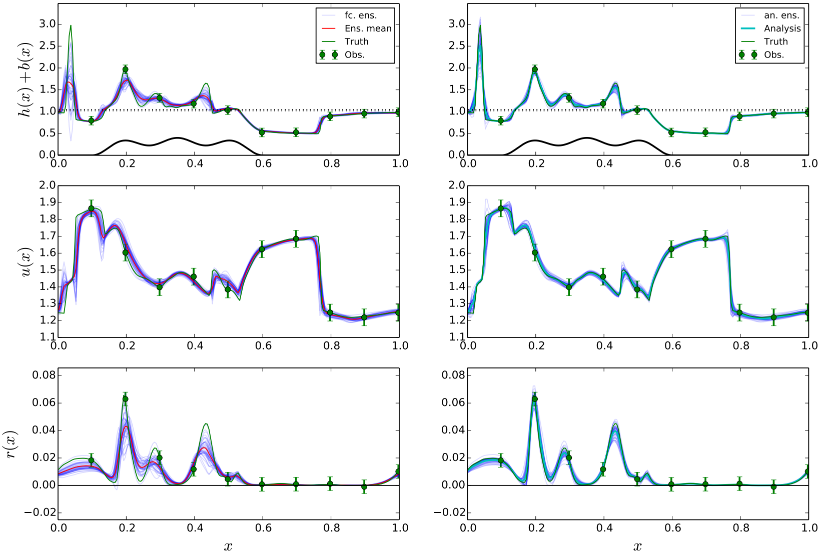
(Excellent) EnKF review paper by Houtekamer and Zhang (2016):
A combination of additive and multiplicative inflation is "critical for maintaining sufficient ensemble spread and good overall performance" ...


Domain-averaged error measures (N = 40):
[od, loc, inf] = [20, 0.0, 1.01]
Ensemble Spread and RMS Error
$(SPR,RMSE)_{an}=(0.0222,0.259)$
$(SPR,RMSE)_{fc}=(0.0257,0.271)$
fc spread
an spread
fc rmse
an rmse
h
$(SPR,RMSE)_{an}=(0.0115,0.0997)$
$(SPR,RMSE)_{fc}=(0.013,0.104)$
u
$(SPR,RMSE)_{an}=(0.000979,0.00982)$
$(SPR,RMSE)_{fc}=(0.00115,0.0104)$
r
Assim. time T

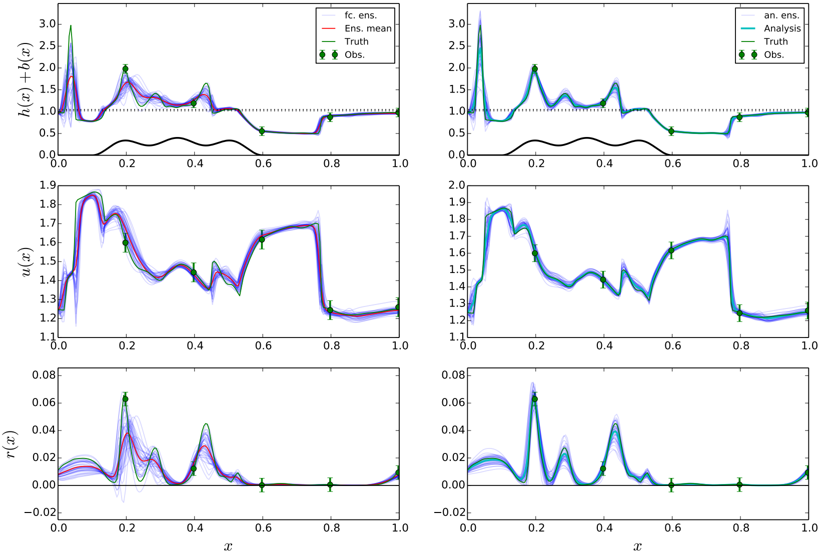

Domain-averaged error measures (N = 40):
[od, loc, inf] = [20, 0.0, 1.01]
Ensemble Spread and RMS Error
$(SPR,RMSE)_{an}=(0.0464,0.0419)$
$(SPR,RMSE)_{fc}=(0.0756,0.132)$
fc spread
an spread
fc rmse
an rmse
h
$(SPR,RMSE)_{an}=(0.018,0.0113)$
$(SPR,RMSE)_{fc}=(0.0247,0.0306)$
u
$(SPR,RMSE)_{an}=(0.0019,0.00167)$
$(SPR,RMSE)_{fc}=(0.00308,0.00513)$
r
Assim. time T


Ensemble trajectories (t = 5.184, N = 40): [od, loc, inf] = [20, 0.0, 1.01]
fc. ens.
Ens. mean
Truth
Obs.
an. ens.
Analysis
Truth
Obs.
$h(x)+b(x)$
$u(x)$
$r(x)$
x
x

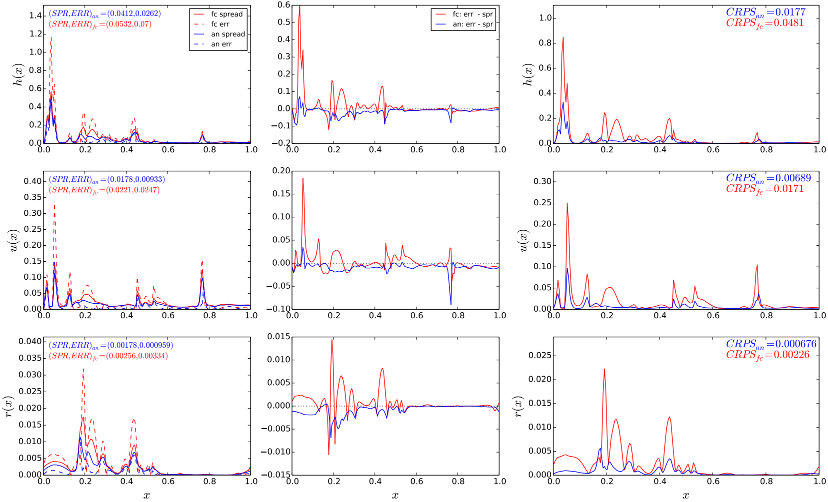
Ensemble trajectories (t = 5.184, N = 40): [od, loc, inf] = [20, 0.0, 1.01]



Ensemble trajectories (t = 5.184, N = 40): [od, loc, inf] = [40, 0.0, 1.01]

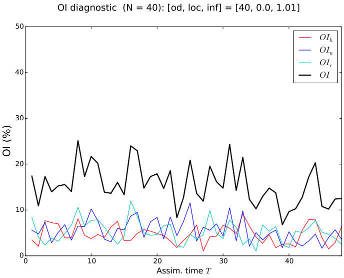


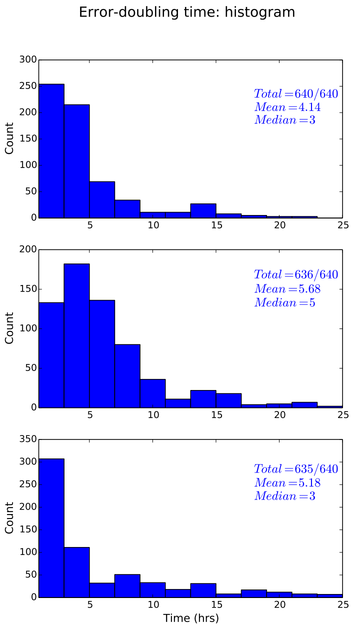
Ensemble spread and error (t = 5.184, N = 40): [od, loc, inf] = [40, 0.0, 1.01]



Relevant for NWP?

- Below: Obs. influence diagnostic (%)
- Right: Error-doubling time statistics





DA: summary

- the idealised fluid model provides an interesting testbed for forecast-assimilation experiments with characteristics relevant for convective-scale NWP
  - sufficient error growth for meaningful hourly-cycled DA at the kilometre–scale
  - varied dynamics throughout with convection and precipitation occuring due to the topographic forcing only (no external forcing required)

DA: summary

- the idealised fluid model provides an interesting testbed for forecast-assimilation experiments with characteristics relevant for convective-scale NWP
  - sufficient error growth for meaningful hourly-cycled DA at the kilometre–scale
  - varied dynamics throughout with convection and precipitation occuring due to the topographic forcing only (no external forcing required)
- EnKF performs adequately (given the simple set-up and strong nonlinearities of the model)
  - certain filter configurations yield ensembles that adequately estimate the forecast errors
  - additive inflation crucial for maintaining satisfactory filter performance
  - no need for localisation since $N > p$ *in the experiments shown* (in fact, localisation degrades the analysis)
  - reasonable (i.e., not too large) multiplicative inflation factors $1.01 - 1.05$
  - the overall observational influence can be tuned to be similar to that of operational NWP

DA: summary and ideas / future work

Numerous experimental set–ups to explore and concepts to investigate...

- comparison of algorithms: how does the EnKF perform against, e.g., a hybrid EnVAR method or a fully nonlinear filter?
- change the way the system is observed, e.g.,
  - observe a subset of the variables only
  - observe nonlinearly (e.g., wind speed and direction)
  - mimic satellite observing systems with a radiative transfer model
- goal: achieve a set-up that requires localisation, e.g., by observing h only

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Plan: integrate the model into EMPIRE (Employing MPI for Researching Ensembles), an open-source repository for interfacing numerical models with DA methods (Browne and Wilson (2015)).

Thanks very much for your attention ...

References:

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- Würsch, M., and Craig, G.C., (2014): A simple dynamical model of cumulus convection for data assimilation research. *Meteorologische Zeitschrift*, 23(5):483-490.

Non-dimensionalised modRSW equations

$$
\begin{aligned}
&\partial_t h + \partial_x (hu) = 0, \\
&\partial_t (hu) + \partial_x (hu^2 + P) + Q\partial_x b + h\widetilde{c_0}^2 \partial_x r - \frac{1}{\mathrm{Ro}} hv = 0, \\
&\partial_t (hv) + \partial_x (huv) + \frac{1}{\mathrm{Ro}} hu = 0, \\
&\partial_t (hr) + \partial_x (hur) + h\widetilde{\beta}\partial_x u + \widetilde{\alpha} hr = 0,
\end{aligned}
$$

where:

$$
\begin{aligned}
P(h,b) &= \frac{1}{2\mathrm{Fr}^2}\left[h^2 + ((H_c - b)^2 - h^2)\Theta(h + b - H_c)\right], \\
Q(h,b) &= \frac{1}{\mathrm{Fr}^2}\left[h + (H_c - b - h)\Theta(h + b - H_c)\right], \\
\widetilde{\beta} &= \beta\Theta(h + b - H_r)\Theta(-\partial_x u).
\end{aligned}
$$

$\Theta(x) = 1$ if $x > 0$; and 0 if $x \leq 0$, and the following parameters are introduced:

$$
\mathrm{Fr} = \frac{V_0}{\sqrt{gH_0}}, \qquad \mathrm{Ro} = \frac{V_0}{fL_0}, \qquad \widetilde{c_0}^2 = \frac{c_0^2}{V_0^2}, \qquad \widetilde{\alpha} = \frac{L_0}{V_0}\alpha.
$$

Some theoretical aspects

- Shallow water systems are hyperbolic, and can thus be solved via a range of numerical recipes for hyperbolic syststems. What about the modified system?

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) + \boldsymbol{G}(\boldsymbol{U})\partial_x \boldsymbol{U} + \boldsymbol{S}(\boldsymbol{U}) = 0,$$

where:

$$\boldsymbol{U} = \begin{bmatrix} h \\ hu \\ hv \\ hr \end{bmatrix}, \boldsymbol{F}(\boldsymbol{U}) = \begin{bmatrix} hu \\ hu^2 + P \\ huv \\ hur \end{bmatrix}, \boldsymbol{G}(\boldsymbol{U}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -c_0^2 r & 0 & 0 & c_0^2 \\ 0 & 0 & 0 & 0 \\ -\widetilde{\beta} u & \widetilde{\beta} & 0 & 0 \end{bmatrix}, \boldsymbol{S}(\boldsymbol{U}) = \begin{bmatrix} 0 \\ Q\partial_x b - fhv \\ fhu \\ \alpha hr \end{bmatrix}$$

- Hyperbolicity determined by eigenstructure (all eigenvalues must be real). Eigenvalues of the system are determined by the Jacobian matrix:

$$\partial \boldsymbol{F}/\partial \boldsymbol{U} + \boldsymbol{G}(\boldsymbol{U}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 - c_0^2 r + \partial_h P & 2u & 0 & c_0^2 \\ -uv & v & u & 0 \\ -u(\widetilde{\beta} + r) & \widetilde{\beta} + r & 0 & u \end{bmatrix},$$

and its four eigenvalues are:

$$\lambda_{1,2} = u \pm \sqrt{\partial_h P + c_0^2 \widetilde{\beta}}, \quad \lambda_{3,4} = u.$$

- Since $P(h;b)$ is non-decreasing and $\widetilde{\beta}$ non-negative, the eigenvalues are real.

DGFEM for modRSW

- TASK: convert the PDE of interest into its equivalent weak formulation using the standard test function and integration approach
- PROBLEM: the presence of NCPs in the governing equations complicates this somewhat because the weak solution in the classical sense of distributions does not exist when the solution becomes discontinuous
- SOLUTION: to overcome the absence of a weak solution, Rhebergen et al. (2008) employ DLM theory (after Dal Maso, LeFloch, and Murat 1995) for NCPs which defines an NCP as a bounded measure in such a way to enable the weak solution to be defined. This is achieved by considering a single NCP $g(u)\partial_x u$, where g is a smooth function but u may admit discontinuities, and defining a smooth regularization u^ϵ of the discontinuous u:

$$g(u)\frac{\mathrm{d}u}{\mathrm{d}x} \equiv \lim_{\epsilon\to 0} g(u^\epsilon)\frac{\mathrm{d}u^\epsilon}{\mathrm{d}x} = C\delta_{x_d}, \text{ with } C = \int_0^1 g(\phi(\tau))\frac{\partial\phi}{\partial\tau}(\tau)\mathrm{d}\tau,$$

  where δ_{x_d} is the Dirac measure at the discontinuity x_d and ϕ is a Lipschitiz continuous path connecting the model states across the discontinuity, an artefact of the regularization.

- The one-dimensional flow domain $\Omega = [0, L]$ is divided into N open elements $K_k = (x_k, x_{k+1})$ for $k = 1, 2, ..., N$ with $N + 1$ nodes/edges $0 = x_1, x_2, ..., x_N, x_{N+1} = L$. Element lengths $|K_k| = x_{k+1} - x_k$ may vary.
- space-DGFEM weak formulation is obtained by (i) multiplying the each equation of the system by an arbitrary test function $w \in C^1(K_k)$, generally continuous on each element but discontinuous across an element boundary; and (ii) integrating (by parts) over each element $K_k \in \mathcal{T}_h$ and summing over all elements. The space discretisation is achieved by replacing the exact model states $\boldsymbol{U}$ and test functions w by approximations $\boldsymbol{U}_h, w_h$ in terms of polynomial basis function expansions, with the order of the polynomials determining the order of the scheme
- In one space dimension and considering cell K_k only at a given t, the weak form reads:

$$0 = \int_{K_k} [w\partial_t U_i - F_i \partial_x w + w G_{ij} \partial_x U_j + w S_i] \, \mathrm{d}x + \left[w(x_{k+1}^L) \mathcal{P}_i^p (\boldsymbol{U}_{k+1}^L, \boldsymbol{U}_{k+1}^R) - w(x_k^R) \mathcal{P}_i^m (\boldsymbol{U}_k^L, \boldsymbol{U}_k^R) \right],$$

where $\mathcal{P}_i^p$ and $\mathcal{P}_i^m$ are the numerical flux terms given by:

$$\mathcal{P}_i^p = \mathcal{P}_i^{NC} + \frac{1}{2} \int_0^1 G_{ij}(\boldsymbol{\phi}) \frac{\partial \phi_j}{\partial \tau} \mathrm{d}\tau,$$
$$\mathcal{P}_i^m = \mathcal{P}_i^{NC} - \frac{1}{2} \int_0^1 G_{ij}(\boldsymbol{\phi}) \frac{\partial \phi_j}{\partial \tau} \mathrm{d}\tau,$$

- ... and the NCP flux through an element edge is:

$$\mathcal{P}_i^{NC}(\boldsymbol{U}^L, \boldsymbol{U}^R) = \begin{cases} F_i^L - \frac{1}{2}\int_0^1 G_{ij}(\boldsymbol{\phi})\frac{\partial \phi_j}{\partial \tau}\mathrm{d}\tau, & \text{if } S^L > 0; \\ F_i^{HLL} - \frac{1}{2}\frac{S^L+S^R}{S^R-S^L}\int_0^1 G_{ij}(\boldsymbol{\phi})\frac{\partial \phi_j}{\partial \tau}\mathrm{d}\tau, & \text{if } S^L < 0 < S^R; \\ F_i^R + \frac{1}{2}\int_0^1 G_{ij}(\boldsymbol{\phi})\frac{\partial \phi_j}{\partial \tau}\mathrm{d}\tau, & \text{if } S^R < 0. \end{cases}$$

- Here, F_i^{HLL} is the standard HLL numerical flux,

$$F_i^{HLL} = \frac{F_i^L S^R - F_i^R S^L + S^L S^R (U_i^R - U_i^L)}{S^R - S^L},$$

  G_{ij} is the ij-th element of the matrix $\boldsymbol{G}$, and $S^{L,R}$ are the fastest left- and right-moving signal velocities in the solution of the Riemann problem, determined by the eigenvalues of the Jacobian of the system:

$$S^L = \min\left(u^L - \sqrt{(\partial_h P)|^L + c_0^2\widetilde{\beta}|^L}, u^R - \sqrt{(\partial_h P)|^R + c_0^2\widetilde{\beta}|^R}\right),$$

$$S^R = \max\left(u^L + \sqrt{(\partial_h P)|^L + c_0^2\widetilde{\beta}|^L}, u^R + \sqrt{(\partial_h P)|^R + c_0^2\widetilde{\beta}|^R}\right).$$

Steady-state modRSW solutions

Consider a system of equations for h, u, and r:

$$\begin{aligned}
&\partial_t h + \partial_x(hu) = 0, \\
&\partial_t u + u\partial_x u + \partial_x \Phi = 0, \\
&\partial_t r + u\partial_x r + \widetilde{\beta}\partial_x u + \alpha r = 0,
\end{aligned}$$

where:

$$\Phi = \begin{cases} \Phi_c + c_0^2 r, & \text{for } h + b > H_c, \\ g(h+b) + c_0^2 r, & \text{otherwise.} \end{cases}$$

Steady-state solutions are found by considering time-independent flow ($\partial_t(\cdot) = 0$):

$$\begin{aligned}
&\partial_x(hu) = 0, \\
&u\partial_x u + \partial_x \Phi = 0, \\
&u\partial_x r + \widetilde{\beta}\partial_x u + \alpha r = 0,
\end{aligned}$$

The first of these steady-state equations gives immediately a solution of u in terms of h:

$$\partial_x(hu) = 0 \implies hu = K, \text{ for constant } K \implies u = \frac{K}{h},$$

which is then substituted into the remaining equations, yielding a system of 2 ODEs to solve for h and r:

$$-\frac{K^2}{h^3}\partial_x h + \partial_x \Phi = 0,$$
$$\frac{K}{h}\partial_x r - \frac{K}{h^2}\widetilde{\beta}\partial_x h + \alpha r = 0.$$

We seek a system of the form $\boldsymbol{M}\boldsymbol{X}' = \boldsymbol{Y}$, where $\boldsymbol{X} = (h, r)^T$, prime denotes derivative with respect to x, and $\boldsymbol{M} \in \mathbb{R}^{2\times 2}$, $\boldsymbol{Y} \in \mathbb{R}^2$ are given from the equations set. If $\boldsymbol{M}$ is non-singular (and hence invertible), then we can solve $\boldsymbol{X}' = \boldsymbol{M}^{-1}\boldsymbol{Y}$ numerically for $\boldsymbol{X}$ using, e.g., a simple finite difference scheme.

The system is expanded as follows:

$$\Big[-\frac{K^2}{h^3}+g|_{H_c}\Big]\partial_x h+\Big[c_0^2\Big]\partial_x r=-\Big[g|_{H_c}\partial_x b\Big],$$
$$\Big[\frac{K}{h}\Big]\partial_x r-\Big[\frac{K}{h^2}\widetilde{\beta}\Big]\partial_x h=-\Big[\alpha r\Big],$$

where $g|_{H_c}=g$ if $h+b\leq H_c$ and zero otherwise and the terms in square brackets are components of $\boldsymbol{M}$ and $\boldsymbol{Y}$:

$$\boldsymbol{M}=\begin{bmatrix}-\frac{K^2}{h^3}+g|_{H_c} & c_0^2\\ -\frac{K}{h^2}\widetilde{\beta} & \frac{K}{h}\end{bmatrix},\quad \boldsymbol{Y}=\begin{bmatrix}-g|_{H_c}\partial_x b\\ -\alpha r\end{bmatrix}.$$

The $\widetilde{\beta}$ term requires further manipulation; re-writing in terms of the Heaviside function we have:

$$\begin{aligned}\widetilde{\beta}&=\beta\Theta(-\partial_x u)\Theta(h+b-H_r)\\&=\beta\Theta(K/h^2\partial_x h)\Theta(h+b-H_r),\\&=\beta\Theta(\partial_x h)\Theta(h+b-H_r).\end{aligned}$$

Thus, the system reads $\boldsymbol{X}'=f(\boldsymbol{X})$ where $f(\boldsymbol{X})=\boldsymbol{M}^{-1}\boldsymbol{Y}$ and is solved using, e.g., a forward Euler finite difference scheme: $\boldsymbol{X}^{j+1}=\boldsymbol{X}^j+\triangle x f(\boldsymbol{X}^j,\boldsymbol{X}^{j-1})$. The value at $j-1$ is required to compute the Heaviside of the height gradient; all other components in $f(\boldsymbol{X})=\boldsymbol{M}^{-1}\boldsymbol{Y}$ are evaluated using values at level j. To start marching through space, note that $\boldsymbol{X}^1=\boldsymbol{X}^2$, so that $\widetilde{\beta}=0$. Then proceed as usual for $j\geq 1$.