Accounting for observation uncertainty due to unresolved scales in data assimilation

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Outline

- Introduction to numerical weather prediction, data assimilation and the error due to unresolved scales
- The Kalman Filter
- Error and bias due to unresolved scales
- The Schmidt-Kalman filter
- Numerical experiments with unbiased and biased observations
- Conclusion
- Future plans

The Numerical Weather Prediction Paradigm

Given an

1. estimate of the current atmospheric state (initial conditions)
2. appropriate description of the boundary conditions

predict future weather using physical laws in the form of PDEs governing the atmospheres evolution.

Data Assimilation

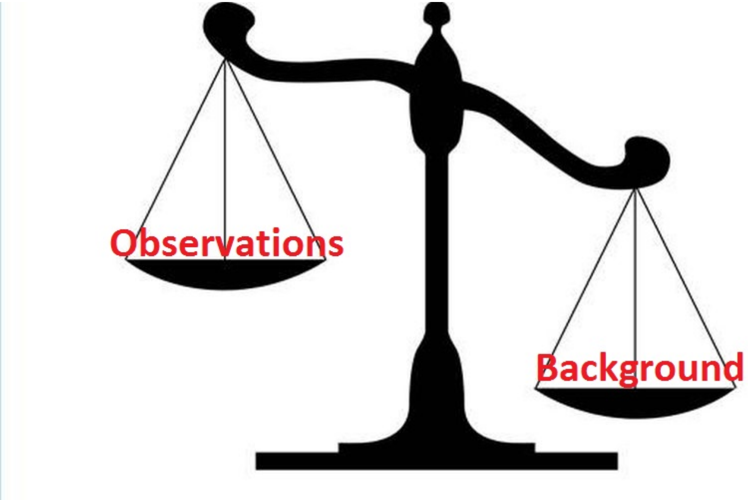
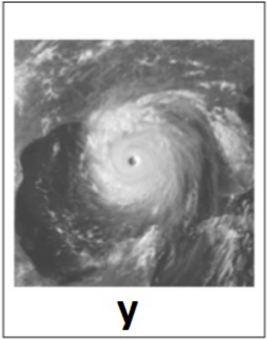


Figure: Data assimilation is about finding the right balance between your observations and your previous forecast.

Error due to Unresolved Scales

The error due to unresolved scales is the difference between a noise-free observation and a model's representation of that observation.



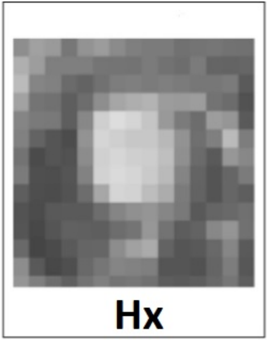
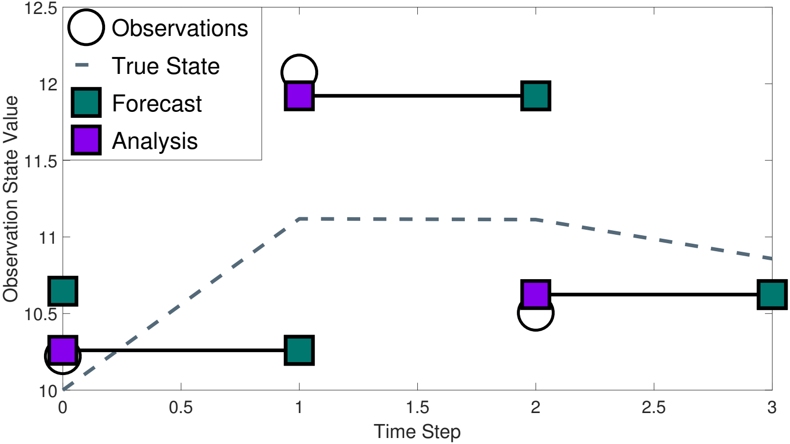


Figure: Janjić et al. (2017)

Aims

1. Demonstrate the effect of the error due to unresolved scales
2. Show how unresolved scales can lead to an observation bias
3. Explain how the Schmidt-Kalman filter can compensate for these errors

The Kalman Filter



The Kalman Filter

Predict: $\boldsymbol{x}_k^f = \mathbf{M}\boldsymbol{x}_{k-1}^a$

Correct: $\boldsymbol{x}_k^a = \boldsymbol{x}_k^f + \boldsymbol{K}_k\left(\boldsymbol{y}_k - \mathbf{H}_k\boldsymbol{x}_k^f\right)$

where $\boldsymbol{K} = \boldsymbol{P}^f\mathbf{H}^T\left(\mathbf{H}\boldsymbol{P}^f\mathbf{H}^T + \boldsymbol{R}\right)^{-1}$,

$\mathbf{M}$ is our model,

$\mathbf{H}$ is the (linearized) observation operator,

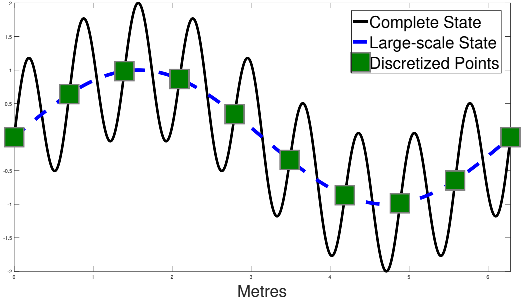
$\boldsymbol{P}^f$ is the forecast error covariance

and $\boldsymbol{R}$ is the observation error covariance.

Error due to Unresolved Scales - Model Structure

We partition our model into large-scale and small-scale components (Janjić & Cohn 2006)

$$\boldsymbol{x}_k^t = \begin{pmatrix} \boldsymbol{x}^{l,t} \\ \boldsymbol{x}^{s,t} \end{pmatrix}_k = \begin{pmatrix} \mathrm{M}^l & \mathbf{0} \\ \mathrm{M}^{sl} & \mathrm{M}^s \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{l,t} \\ \boldsymbol{x}^{s,t} \end{pmatrix}_{k-1} + \begin{pmatrix} \boldsymbol{\eta}^l \\ \boldsymbol{\eta}^s \end{pmatrix}_k$$



Error due to Unresolved Scales - Observations

We now consider observations to be a combination of large-scale and small-scale components plus instrument error ϵ:

$$\boldsymbol{y} = \mathbf{H}^{l,t}\boldsymbol{x}^{l,t} + \mathbf{H}^{s,t}\boldsymbol{x}^{s,t} + \boldsymbol{\epsilon}.$$

The representation error is then given by

$$\boldsymbol{\xi} = \begin{pmatrix} \mathbf{H}^{l,t} - \mathbf{H}^{m} & \mathbf{H}^{s,t} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{l,t} \\ \boldsymbol{x}^{s,t} \end{pmatrix},$$

where $\mathbf{H}^m$ is the (possibly incorrect) observation operator used in our filter.

Bias due to Unresolved Scales

Dropping the t superscripts and assuming $\mathbf{M}^{sl}$ and $\mathbf{M}^{s}$ are perfect the true small-scale state is given by

$$\boldsymbol{x}_k^s = \mathbf{M}^{sl}\boldsymbol{x}_{k-1}^l + \mathbf{M}^s\boldsymbol{x}_{k-1}^s.$$

Assuming no observation operator error we will have

$$\begin{aligned}
\langle \boldsymbol{\xi}_k \rangle &= \langle \mathbf{H}^s(\boldsymbol{x}_k^s) \rangle \\
&= \langle \mathbf{H}^s(\mathbf{M}^{sl}\boldsymbol{x}_{k-1}^l + \mathbf{M}^s\boldsymbol{x}_{k-1}^s) \rangle \\
&= \langle \mathbf{H}^s(\underline{\mathbf{M}^{sl}\boldsymbol{x}_{k-1}^l} + \mathbf{M}^s(\ldots(\underline{\mathbf{M}^{sl}\boldsymbol{x}_0^l} + \mathbf{M}^s(\boldsymbol{x}_0^s))))\rangle.
\end{aligned}$$

So potentially $\langle \boldsymbol{\xi}_k \rangle \neq 0$.

The Schmidt-Kalman Filter (Schmidt 1966)

Use the statistics of the large-scale and small-scale processes:

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}^{ll} & \boldsymbol{P}^{ls} \\ \boldsymbol{P}^{sl} & \boldsymbol{P}^{ss} \end{pmatrix}.$$

Analysis updates with modelled small-scale error covariance $\boldsymbol{C}^s$ (Janjić & Cohn 2006):

$$\begin{aligned}
\boldsymbol{x}^{l,a} &= \boldsymbol{x}^{l,f} + \boldsymbol{K}^l(\boldsymbol{y} - \mathbf{H}^l \boldsymbol{x}^{l,f}), \\
\boldsymbol{K}^l &= \left(\begin{pmatrix} \boldsymbol{P}^{ll,f} & \boldsymbol{P}^{ls,f} \end{pmatrix} \begin{pmatrix} (\mathbf{H}^l)^T \\ (\mathbf{H}^s)^T \end{pmatrix} \right) \boldsymbol{D}^{-1} \\
\boldsymbol{D} &= \mathbf{H}^l \boldsymbol{P}^{ll,f} (\mathbf{H}^l)^T + \mathbf{H}^l \boldsymbol{P}^{ls,f} (\mathbf{H}^s)^T \\
&\quad + \mathbf{H}^s \boldsymbol{P}^{sl,f} (\mathbf{H}^l)^T + \mathbf{H}^s \boldsymbol{C}^s (\mathbf{H}^s)^T + \boldsymbol{R}.
\end{aligned}$$

Evaluating the True Analysis Error of the Schmidt-Kalman Filter

When deriving the true analysis error equation for the Schmidt-Kalman filter we make the following assumptions.

1. All models and observation operators are linear.
2. We have exact knowledge of what the small-scales are.
3. We filter the large-scale state only.

True Analysis Error Equation

$$\begin{pmatrix} \boldsymbol{e}_k^{l,a} \\ \boldsymbol{e}_k^{s,a} \end{pmatrix} = \left[\boldsymbol{I} - \begin{pmatrix} \boldsymbol{K}_k^l \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{H}^l & \mathbf{H}^s \end{pmatrix} \right] \begin{pmatrix} \mathbf{M}^l & \mathbf{0} \\ \mathbf{M}^{sl} & \mathbf{M}^s \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_{k-1}^{l,a} \\ \boldsymbol{e}_{k-1}^{s,a} \end{pmatrix}$$

$$+ \left[\boldsymbol{I} - \begin{pmatrix} \boldsymbol{K}_k^l \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{H}^l & \mathbf{H}^s \end{pmatrix} \right] \begin{pmatrix} \boldsymbol{\eta}_k^l \\ \boldsymbol{\eta}_k^s \end{pmatrix}$$

$$+ \begin{pmatrix} \boldsymbol{K}_k^l \\ \mathbf{0} \end{pmatrix} \left(\gamma_k + \boldsymbol{\epsilon}_k + \mathbf{H}^s \boldsymbol{x}_k^{s,f} \right)$$

- $\boldsymbol{e}^{l,a}, \boldsymbol{e}^{s,a} \rightarrow$ state errors
- $\boldsymbol{\eta}^l, \boldsymbol{\eta}^s \rightarrow$ model errors
- $\gamma \rightarrow$ observation operator error
- $\epsilon \rightarrow$ instrument error

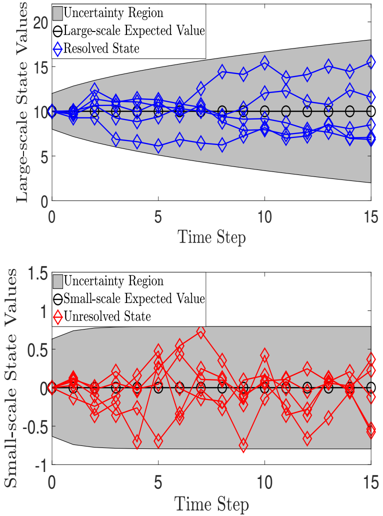
Numerical Experiments - Random Walk Model

$$\begin{pmatrix} x^l \\ x^s \end{pmatrix}_{k+1} = \begin{pmatrix} 1 & 0 \\ \alpha & e^{-1/2} \end{pmatrix} \begin{pmatrix} x^l \\ x^s \end{pmatrix}_k + \begin{pmatrix} \eta^l \\ \eta^s \end{pmatrix}_k ,$$

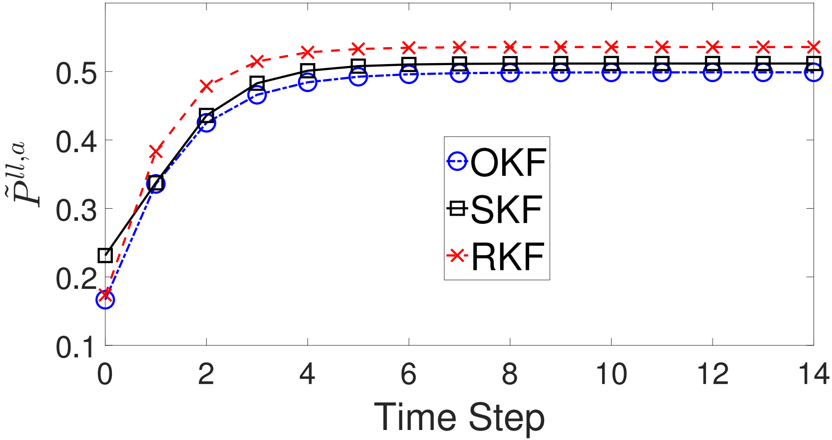
$$y_k = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x^l \\ x^s \end{pmatrix}_k + \epsilon_k$$

where $\eta_k^l \sim \mathcal{N}(0,1)$, $\eta_k^s \sim \mathcal{N}(0,0.3)$ and $\epsilon_k \sim \mathcal{N}(0,0.1)$.

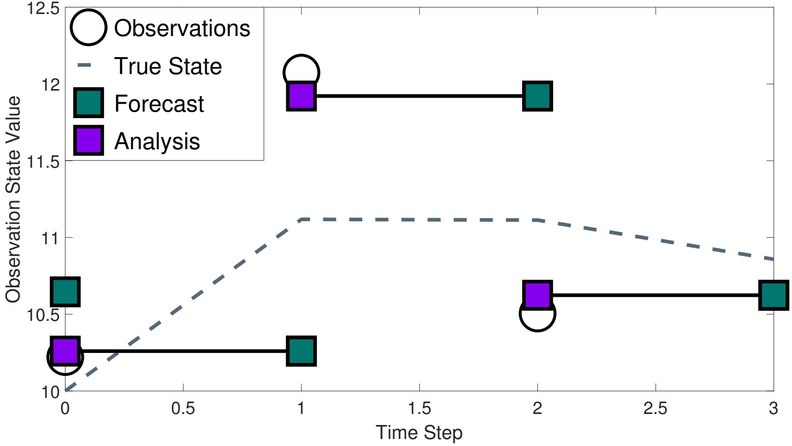
For unbiased observations $\alpha = 0$.



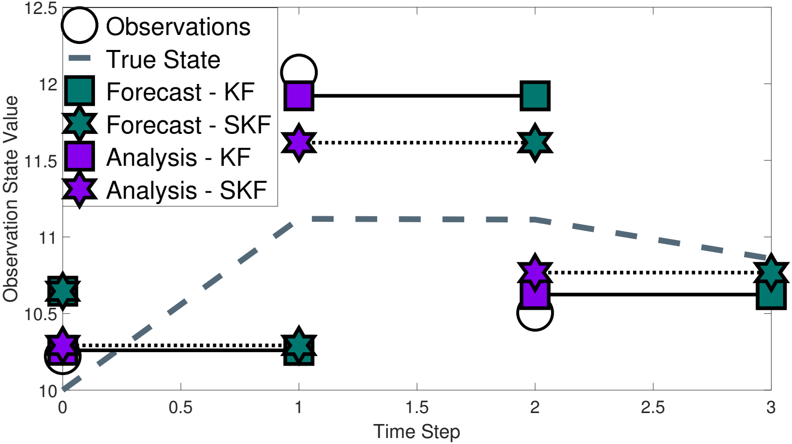
Numerical Experiments - Large-scale Analysis Error Variances



The Reduced Kalman Filter



The Schmidt-Kalman Filter



Bias Correction

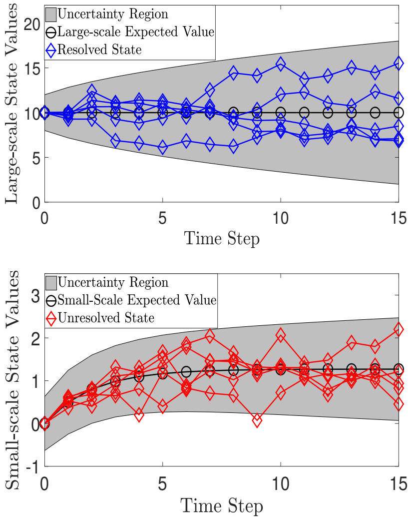
Introducing an Observation Bias due to Unresolved Scales

$$\begin{pmatrix} x^l \\ x^s \end{pmatrix}_{k+1} = \begin{pmatrix} 1 & 0 \\ \alpha & e^{-1/2} \end{pmatrix} \begin{pmatrix} x^l \\ x^s \end{pmatrix}_k + \begin{pmatrix} \eta^l \\ \eta^s \end{pmatrix}_k ,$$

$$y_k = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x^l \\ x^s \end{pmatrix}_k + \epsilon_k$$

where $\eta_k^l \sim \mathcal{N}(0,1)$, $\eta_k^s \sim \mathcal{N}(0,0.3)$ and $\epsilon_k \sim \mathcal{N}(0,0.1)$.

For biased observations $\alpha \neq 0$.
(We take $\alpha = 0.05$)



Observation Bias Correction with State Augmentation

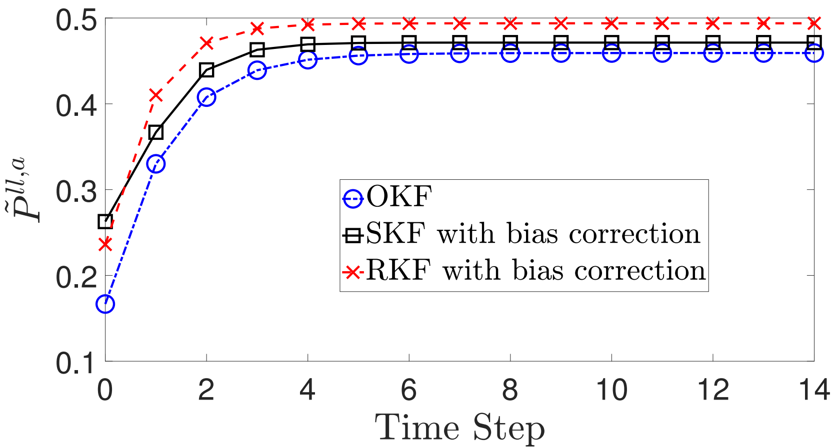
Correct observation bias ⇒ Augment the state (Friedland 1969, Jazwinski 2007)

We assume we can express our model in the form

$$\begin{pmatrix} \boldsymbol{x}^{l,t} \\ \boldsymbol{x}^{\beta,t} \\ \boldsymbol{x}^{\delta,t} \end{pmatrix}_k = \begin{pmatrix} \mathrm{M}^l & \mathbf{0} & \mathbf{0} \\ \mathrm{M}^{sl} & \mathrm{M}^s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{M}^s \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{l,t} \\ \boldsymbol{x}^{\beta,t} \\ \boldsymbol{x}^{\delta,t} \end{pmatrix}_{k-1} + \begin{pmatrix} \boldsymbol{\eta}^l \\ \mathbf{0} \\ \boldsymbol{\eta}^s \end{pmatrix}_k ,$$

where $\boldsymbol{x}^{\beta}$ is the bias state and $\boldsymbol{x}^{\delta}$ is the unbiased unresolved state.
$(\boldsymbol{x}^s = \boldsymbol{x}^{\beta} + \boldsymbol{x}^{\delta})$

Numerical Experiments for Bias Correcting Filters - Large-scale Analysis Error Variance



Conclusion

1. The error due to unresolved scales is a consequence of not being able to completely represent an observable process.
2. The Schmidt-Kalman filter can compensate for error due to unresolved scales with our simple model.
3. We can use state augmentation to correct observation biases.
4. The Schmidt-Kalman filter is compatible with this method of bias correction.

Current and Future Work

Now starting to examine car temperature data by

1. examining the quality of the data
2. looking into the natural variability of the data
3. looking into representation errors in a real meteorological data set

Future work for the Schmidt-Kalman filter includes

1. adapting this filter to nonlinear models (i.e. ensemble SKF, variational or ensemble-variational form)
2. determine a suitable way to model/approximate the small-scale error covariance
3. mitigating the computational cost associated with this filter

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